Global non-equilibrium attractor for non-linear Fokker-Planck systems

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Stochastic processes are ubiquitous in physics. Systems of interest are usually not isolated but in contact with a much larger environment. What makes their dynamics stochastic is the interaction with the environment (thermal bath): the integration over its degrees of freedom entails that the "force"—understood in a generalised sense—acting on the system becomes effectively random. It is in this approach, often called mesoscopic, that the Langevin or Fokker-Planck equations emerge.

Let us consider a physical system whose mesoscopic states are described by $\mathbf{r} \equiv \{x_1, \ldots, x_d\}$. A prototypical example is a colloidal particle confined in a d-dimensional potential well. We assume the dynamics of $\mathbf{r}$ to be Markovian and governed by the following "fully non-linear" Fokker-Planck equation for the probability distribution function $P = P(\mathbf{r}, t)$ in the ensemble picture [1],

$$\partial_t P = \nabla_{\mathbf{r}} \cdot \left[\mathbf{A}(\mathbf{r}) P + \frac{1}{2} B^2(\mathbf{r}) \nabla_{\mathbf{r}} P \right]. \tag{1}$$

Our terming "fully non-linear" for the Fokker-Planck equation stresses the fact that, in general, not only the "force" $\mathbf{A}(\mathbf{r})$ but also the diffusivity $B^2(\mathbf{r})$ are non-linear functions of $\mathbf{r}$. The dynamics of the system is stochastic due to its contact with a thermal bath at temperature T. We assume that detailed balance holds, so the fluctuation-dissipation relation

$$2\mathbf{A}(\mathbf{r}) = \beta\, B^2(\mathbf{r})\, \nabla H(\mathbf{r}). \tag{2}$$

is verified, $H(\mathbf{r})$ being the system's "Hamiltonian" and $\beta = (k_B T)^{-1}$. In certain contexts, $H(\mathbf{r})$ would not be the Hamiltonian of the system but the function playing its role: e.g., for an overdamped Brownian particle, $H(\mathbf{r})$ would be the confining potential, with $\mathbf{r}$ being the spatial coordinates, while for a molecular fluid within the context of kinetic theory, $H(\mathbf{r})$ would be the kinetic energy, with $\mathbf{r}$ now accounting for the velocities of the particles of the fluid [2]. Eq. (2) entails that the canonical distribution, proportional to $e^{-\beta H(\mathbf{r})}$, is the stationary solution of the Fokker-Planck equation.

In the long-time limit, systems evolving under stochastic dynamics typically relax to equilibrium at the bath temperature. The equilibrium state is thus a global attractor, reached from an arbitrary initial condition, of the system dynamics. A relevant question is whether it is only the final equilibrium state that is independent of the initial preparation or there appears a previous global non-equilibrium attractor, already independent of the initial preparation. In the latter case, relaxation to equilibrium would proceed in two stages: first, the system would approach the universal non-equilibrium state and, second, this non-equilibrium state would tend to equilibrium.

Here we show—under general assumptions—that there emerges such a universal non-equilibrium state for a wide class of systems described by a non-linear Langevin equation, when quenched to low enough temperatures. This state, which we term long-lived non-equilibrium state (LLNES), is a global attractor of the dynamics for an intermediate time scale, over which initial conditions are already forgotten but the system is still far from equilibrium. In particular, the probability distribution function exhibits a Dirac-delta shape within the LLNES [1], as shown in Fig.1.

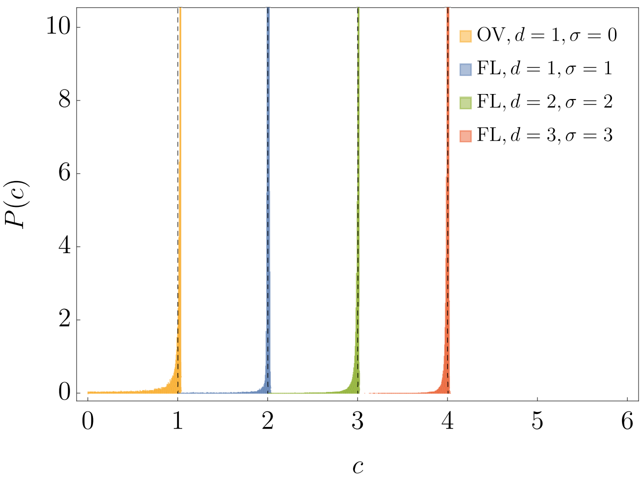

Fig. 1. Scaled probability distribution function at the LLNES for different physical situations. Plots for both (OV) an overdamped particle in a non-harmonic potential and (FL) a molecular fluid with non-linear drag, in different dimensions. For the former, $\mathbf{c} = \mathbf{r}/\langle r \rangle$ stands for the scaled spatial position of the confined overdamped particle; while for the latter, $\mathbf{c} = \mathbf{v}/\langle v \rangle$ accounts for the scaled velocities of the particles of the molecular fluid. In order to appreciate the universal Dirac-delta shape, each probability distribution function is shifted an amount σ to the right, as indicated in the legend.

[1] A. Patrón, B. Sánchez-Rey, E. Trizac and A. Prados, *Non-equilibrium attractor for non-linear stochastic dynamics*, Phys. Rev. Lett. (Under review)

[2] A. Patrón, B. Sánchez-Rey, and A. Prados, *Strong non-exponential relaxation and memory effects in a fluid with non-linear drag*, Phys. Rev. E **104**, 064127 (2021).