D-Mercator: multidimensional hyperbolic embedding of real networks

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Summary

One of the pillars of the geometric approach to networks has been the development of model-based mapping tools that embed network topologies in their latent geometry. In particular, Mercator [1] embeds real networks into the hyperbolic plane. However, some real networks are better described by the multidimensional formulation of the underlying geometric model [2]. Here, we introduce D-Mercator [3], an embedding method that goes beyond Mercator to produce multidimensional maps of real networks into the $D+1$ hyperbolic space where the similarity dimension is represented in a D-sphere. We evaluated the quality of the embeddings using synthetic S^D networks. We also produced multidimensional hyperbolic maps of real networks that provide more informative descriptions than their two-dimensional counterparts and reproduce their structure more faithfully. Having multidimensional representations will help to reveal the correlation of the dimensions identified with factors known to determine connectivity in real systems and to address fundamental issues that hinge on dimensionality, such as universality in critical behavior. D-Mercator also allows us to estimate the intrinsic dimensionality of real networks in terms of navigability and community structure, in good agreement with embedding-free estimations.

Multidimensional network model

Our approach assumes that real networks are well described by the geometric soft configuration model in D similarity dimensions, the S^D/H^{D+1} model [4], which is a multidimensional generalization of the S^1 model. In the S^D model, a node i is endowed with a hidden variable representing its popularity, influence, or importance, denoted κ_i and named hidden degree, and with a position $\mathbf{v}_i$ in the D-dimensional similarity space, represented as a vector in a D-dimensional sphere. The connection probability between a node i and a node j takes the form of a gravity law:

$$p_{ij} = \frac{1}{1+\chi_{ij}^{\beta}}, \quad \text{with} \quad \chi_{ij} = \frac{R\Delta\theta_{ij}}{(\mu\kappa_i\kappa_j)^{1/D}}. \qquad (1)$$

The separation $\Delta\theta_{ij} = \arccos(\frac{\mathbf{v}_i \cdot \mathbf{v}_j}{||\mathbf{v}_i||||\mathbf{v}_j||})$ represents the angular distance between nodes i and j in the D-dimensional similarity space. The parameter β, named inverse temperature, calibrates the coupling of the network topology with the underlying metric space and controls the level of clustering, which grows with the increase of β. Finally, the parameter μ controls the average degree of the network.

Multidimensional maps of real world networks

Real networks can be embedded in any dimensions within the limits of D-Mercator. We compiled data for several real-world complex networks from different domains and embedded them in different dimensions. More specifically, here we present Add-health [5] network, which describes the friendships between high school students. In this case, the best embedding dimension is $D=2$ such that the similarity subspace can be easily visualized in three dimensions as a 2-sphere.

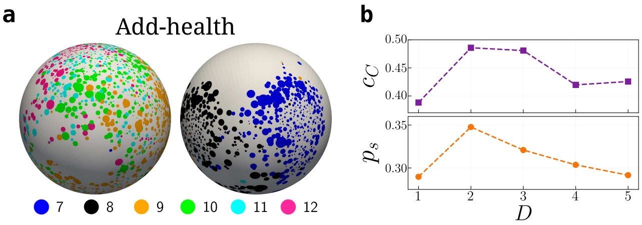

Fig. 1. Embeddings of Add-health network. **(a)** Two views of the D-Mercator embedding in $D=2$. The size of a node is proportional to its expected degree. Nodes are colored based on their communities, i.e., a grade the student belongs to. Panel **(b)** shows the performance of geometric community concentration c_C and the success rate of greedy routing (p_s) in different embedded dimension.

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