Explosive synchronization of higher-order Kuramoto oscillators

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From social interactions to the human brain, networks and their high-order counterpart are key to describing the underlying structure and dynamics of complex systems. While it is well known that network structure strongly affects its function, the role that the underlying geometry and topology play on the emergent dynamical properties of a system is yet to be clarified (figure 1. Here I consider the synchronization of higher-order topological signals, associated not only with the nodes of a system but in general with any of its higher-dimensional components or simplices: links, triangles, tetrahedra, and so on. Whereas nodes interact only through links, higher-order simplices in general may interact both through their lower or higher dimensional faces. For instance, links may interact through the nodes and also the triangles that they share. These interactions are captured by the $n-$dimensional ($n > 1$) Hodge Laplacian $\mathcal{L}_n$:

$$\mathcal{L}_n = \mathcal{B}_n^\top \mathcal{B}_n + \mathcal{B}_{n+1}\mathcal{B}_{n+1}^\top = \mathcal{L}_n^{\text{up}} + \mathcal{L}_n^{\text{down}}, \tag{1}$$

where $\mathcal{B}_n$ is the incidence matrix projecting a simplicial complex of dimension n to its $n-1$-dimensional boundary, and $\mathcal{L}_n^{\text{up}}$ and $\mathcal{L}_n^{\text{down}}$ encode respectively the interaction through the upper and lower dimensional faces.

Following the definition of the Hodge Laplacian, we propose the generalization of the Kuramoto dynamics including coupling between higher-order simplicies as [1]:

$$\hat{\theta} = \omega - \sigma \mathcal{B}_{n+1} \sin \mathcal{B}_{n+1}^\top \theta - \sigma \mathcal{B}_n^\top \sin \mathcal{B}_n \theta, \tag{2}$$

where θ and ω indicate respectively the vectors of phases and intrinsic frequencies associated with each simplex. By using the Hodge decomposition (separating a vector into its harmonic, irrotational and solenoidal components), one can see that the projections of θ on the upper and lower dimensional phases, i.e. $\theta^{[+]} = \mathcal{B}_{n+1}^\top \theta$ and $\theta^{[+]} = \mathcal{B}_n \theta$, decouple. Remarkably, the dynamics on both of these projections displays a continuous phase transition on the associated Kuramoto order parameter, R^+ and R^-, with a critical point $\sigma_c = 0$, as shown in figure 2 (blue points).

We propose an alternative definition of the higher-order Kuramoto model, in which now the dynamics associated with the lower and higher dimensional faces are coupled:

$$\hat{\theta} = \omega - \sigma R^- \mathcal{B}_{n+1} \sin \mathcal{B}_{n+1}^\top \theta - \sigma R^+ \mathcal{B}_n^\top \sin \mathcal{B}_n \theta. \tag{3}$$

In this formulation, the higher-order Kuramoto dynamics displays projected on the upper and lower dimensional phases display an explosive phase transition, as shown in figure 2 (orange points).

In summary, the higher-order Kuramoto dynamics can lead to either a continuous or explosive synchronization transition, depending on the existence or not of coupling between the solenoidal and irrotational components of the dynamics. The topological transition cannot be observed via the standard Kuramoto order parameter, but it requires the application of *topological filters* to retain only the solenoidal or irrotational components of the data.

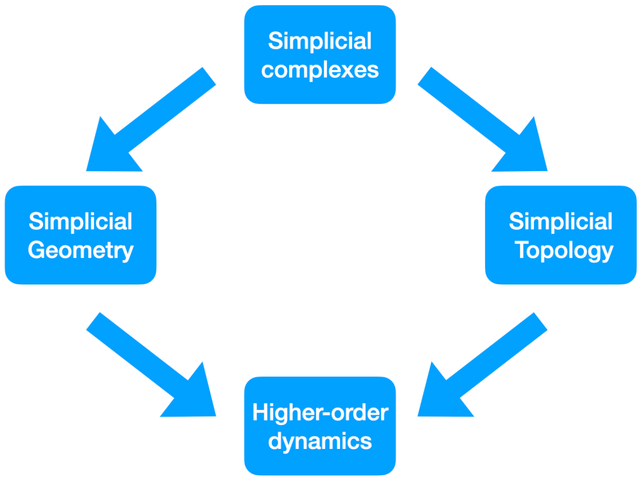

Fig. 1. Simplicial complexes encode the geometry and topology of the data, which strongly affect higher-order dynamics. Figure extracted from [2].

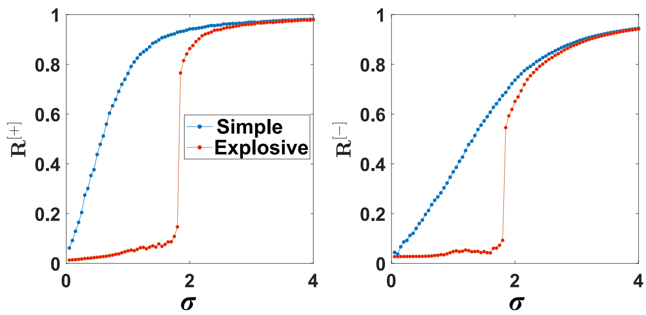

Fig. 2. The order parameters R^+ and R^- reveal the synchronization transition of topological signals defined on links ($n = 1$) and coupled with the continuous (blue line) and explosive (red line) versions of the higher-order Kuramoto model, respectively.

[1] Ana P. Millán, Joaquín J. Torres and Ginestra Bianconi, *Explosive higher-order Kuramoto dynamics on simplicial complexes*, Physical Review L **124**.21, 218301 (2021).

[2] Ana P. Millán, Juan G. Restrepo, Joaquín J. Torres and Ginestra Bianconi, *Geometry, topology and simplicial synchronization* in *Higher-Order Systems* (pp. 268-299) (Springer International Publishing, 2022).