Sampling rare trajectories in stochastic processes

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Despite their infrequent occurrence, rare events in stochastic processes can lead to the most catastrophic outcomes. Examples where an unlikely combination of events with a large impact occurs include natural disasters such as earthquakes or volcanic eruptions, stock market crashes, fade-outs of epidemics and species extinctions. Much interest has recently been focused on the sampling of rare trajectories and the quantification of their statistics in models of stochastic phenomena. This problem is computationally demanding if conventional sampling methods are used, so specific rare-trajectory sampling techniques must be developed to deal with it.

The renowned Wentzel-Kramers-Brillouin (WKB) method constitutes a tool to characterise most likely paths describing rare events, but it is only valid in the limit of small noise. A recently proposed backtracking sampling method that overcomes this limitation consists of working with so-called stochastic bridges, which are trajectories that are constrained to have fixed start and end points. The novelty of this method is that stochastic bridges are generated backwards in time [1]. The aim of our work is to employ the backtracking method to sample rare trajectories in stochastic models belonging to a wide variety of disciplines. We are specially interested in the study of problems with absorbing states, from which the system cannot escape once it reaches them.

To illustrate the power of the backtracking method, let us focus on the escape of a Brownian particle from the potential $U(x) = (x^2-1)^2/4$, which has two minima at $x = \pm 1$ separated by a barrier of height $1/4$. The deterministic dynamics of the particle is governed by:

$$\frac{dx(t)}{dt} = -\frac{\partial U(x)}{\partial x}. \tag{1}$$

This system has three fixed points: the wells $x = \pm 1$ (attracting) and the origin (repelling). Transitions of the particle from one well to the other passing through the intermediate repelling point $x = 0$ are forbidden in the deterministic limit, but become possible if stochasticity is taken into account. In such case, the mean time required for the particle to escape from one well increases exponentially with the inverse noise strength of the system. For small noise amplitudes D it is therefore difficult to observe escape paths from one well to the other in direct simulations of the stochastic process. Figure 1 shows a set of stochastic bridges, constructed using the backtracking method, representing such rare trajectories. On the other hand, the WKB formalism is incapable of describing the most likely path connecting the two wells, but only characterises the optimal transition path from one well to the repelling state $x = 0$, as can be seen in Figure 2. This figure also shows a set of stochastic bridges at finite noise amplitude, which fluctuate around the WKB most probable path valid in the small-noise limit.

Our current research focuses on further exploiting the backtracking method, which we propose as a simpler and more comprehensive method for the sampling of rare trajectories to those already existing in the literature. On the other hand, we are exploring the backtracking method as a technique to study the influence of the nature of the noise on the transition paths between different states in rare events. In particular, we want to determine whether quantities such as jump times between states or path fluctuations are affected by the type of noise considered.

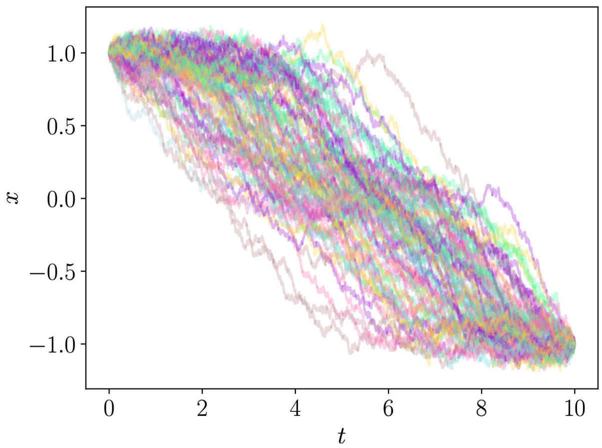

Fig. 1. Sample of 100 transition paths from one well to the other of a Brownian particle moving in a quartic double-well potential.

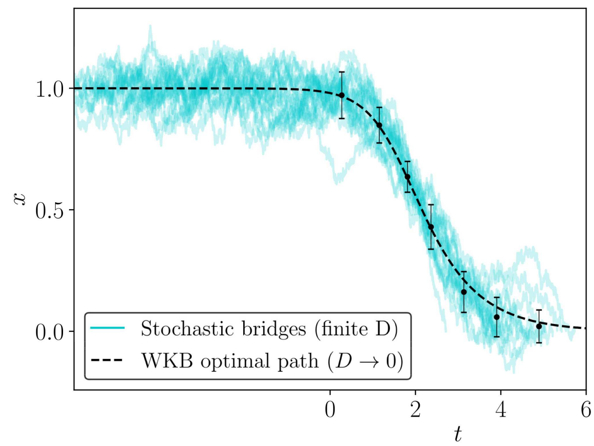

Fig. 2. WKB most likely transition path from one well to the origin, valid at the small-noise limit, together with a set of stochastic bridges at finite noise amplitude.

[1] J. Aguilar, J. W. Baron, T. Galla and R. Toral, *Sampling rare trajectories using stochastic bridges*, Physical Review E **105**, 064138 (2022).