Non-equilibrium criticality in the synchronization of self-sustained oscillators

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Synchronization is a pervasive collective phenomenon frequently displayed by systems of scientific interest across disciplines, from neurons and circadian rhythms to lasers and qubits [1]. The study of synchronous dynamics has traditionally focused on the identification of threshold parameter values for the transition to synchronization, and on the nature of such transition. The dynamical process whereby systems of self-sustained oscillators synchronize, however, has been devoted much less attention. The fact that one might reasonably expect this process to be strongly system-dependent probably explains why such an important aspect of synchronization remains poorly understood.

In a recent contribution [2], we have shown that, in fact, the synchronization process displays robust universal features, some of which have been previously studied in the literature on kinetic roughening of interfaces. This is another active branch of contemporary statistical physics that studies universal properties in non-equilibrium growth processes, covering also a great diversity of cases, including, for instance, coffee ring formation, bacterial growth and natural and artificial deposition processes [3]. The overlap between the two fields originates in a mathematical connection between synchronization models and the equations of surface growth, which has been discussed in the literature [1], but whose dynamical implications do not seem to have been investigated until now. Indeed, starting form a system of phase oscillators (i.e. idealized dissipative dynamical systems with an attracting limit cycle) at the sites of a lattice, and performing a continuum approximation, for a relatively slow spatial variation of the phase field $\phi(\mathbf{x}, t)$, as occurs for coupling strengths well into the synchronized regime, the dominant contributions yield

$$\partial_t \phi(\mathbf{x}, t) = \omega^*(\mathbf{x}) + \nu \nabla^2 \phi(\mathbf{x}, t) + \frac{\lambda}{2} [\nabla \phi(\mathbf{x}, t)]^2. \qquad (1)$$

Eq. (1), which features the same deterministic derivative terms as the standard Kardar-Parisi-Zhang (KPZ) equation, is known the KPZ equation with columnar noise [4].

In Ref. [2], by means of a detailed numerical study of one-dimensional systems of phase oscillators coupled through a function containing just one Fourier component, we provide strong evidence indicating that the synchronization process in these systems is characterized by forms of generic scale invariance associated with the universality classes of kinetically rough interfaces with columnar disorder. Specifically, for Kuramoto coupling the relevant universality class is that of the Edwards-Wilkinson equation with columnar noise, while for generic couplings it is that of the KPZ equation with columnar noise. Moreover, the phase fluctuations around the average growth follow a ubiquitous Tracy-Widom (TW) probability distribution, which is frequently associated with the KPZ nonlinearity, in the latter case, see Fig. 1. For the highly-symmetric Kuramoto case, the fluctuations are simply Gaussian.

More recent results will be discussed where this connection is shown to be generalized in a straighforward way to phase oscillators with an arbitrary number of harmonics, and also to actual two-dimensional systems featuring a stable limit cycle in their phase space, including Stuart-Landau oscillators and van der Pol oscillators. As in general in such systems the odd symmetry of the Kuramoto (sine) coupling never holds, the critical behavior is consistently that of the KPZ equation with columnar noise, with TW flucttutations, in agreement with the observation made in Ref. [2]. Synchronization and surface growth processes with columnar noise thus seem to be much more closely related than previously anticipated; they are both instances of generic scale invariance with anomalous scaling forms.

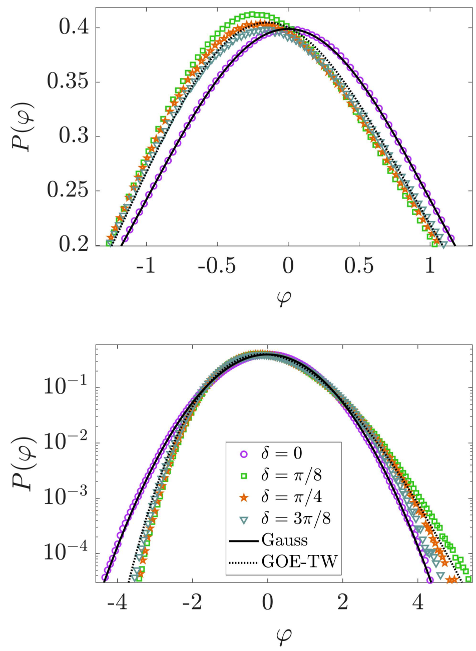

Fig. 1. Fluctuations around the average growth of the synchronization process for Kuramoto coupling ($\delta = 0$), which follows a Gaussian distribution, and for other (non-odd) couplings ($\delta \neq 0$), which follow a TW distribution.

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[4] I. G. Szendro, J. M. López and M. A. Rodríguez, *Localization in disordered media, anomalous roughening, and coarsening dynamics of faceted surfaces*, Phys. Rev. E **76**, 011603 (2007).